For Online Publication: Appendix

Proof of Proposition 1. The solution strategy is to design an incentive-compatible mechanism $(P(\hat{f}), \theta(\hat{f}))$ which maximizes the seller's profit and satisfies the buyer's free-entry condition. Note that for the buyer's free-entry condition to be satisfied under a fully separating mechanism, we must have:

$$P(\hat{f}) = \hat{f} - \frac{k\theta(\hat{f})}{m(\theta(\hat{f}))}.$$

Using the above expression for P, I can write the seller's payoff for a given mechanism θ as

$$\begin{aligned} U(\hat{f}|f) &= m(\theta(\hat{f}))\left(P(\hat{f}) - \delta f\right) \\ &= m(\theta(\hat{f}))\left(\hat{f} - \delta f\right) - k\theta(\hat{f}), \end{aligned}$$

so that $U(\hat{f}|f)$ is the payoff of a type f seller who reports $\hat{f}$. Define $\bar{U}(f) \equiv U(f|f)$ to be the profit to a seller who reports the truth, given a particular mechanism $\theta(\hat{f})$. The following lemma gives necessary and sufficient conditions for global incentive compatibility.

Lemma 1. *Global incentive compatibility (GIC), defined as*

$$U(f|f) \geq U(\hat{f}|f) \quad \forall f, \hat{f} \in [\underline{f}, \bar{f}]$$

is equivalent to

(i) Local incentive compatibility (LIC): $\bar{U}'(f) = U_2(f|f)$ *or* $U_1(f|f) = 0$ *almost surely, and*

(ii) Monotonicity (M): $U_{21}(\hat{f}|f) \geq 0$ *almost surely.*

Proof. Global incentive compatibility implies

$$\frac{U(f|f) - U(\hat{f}|\hat{f})}{f - \hat{f}} \geq \frac{U(\hat{f}|f) - U(\hat{f}|\hat{f})}{f - \hat{f}}$$

if $f > \hat{f}$, and the sign reversed if $f < \hat{f}$. Letting $f \to \hat{f}$ from above and below immediately gives local incentive compatibility.

Now rewrite (GIC) as follows:

$$\begin{aligned}
0 &\leq U(f|f) - U(\hat{f}|f) \\
&= \Big(U(f|f) - U(\hat{f}|\hat{f})\Big) - \Big(U(\hat{f}|f) - U(\hat{f}|\hat{f})\Big) \\
&= \int_{\hat{f}}^{f} \bar{U}'(s)ds - \int_{\hat{f}}^{f} U_2(\hat{f}|s)ds \\
&= \int_{\hat{f}}^{f} U_2(s|s)ds - \int_{\hat{f}}^{f} U_2(\hat{f}|s)ds \qquad (from \quad LIC) \\
&= \int_{\hat{f}}^{f} \int_{\hat{f}}^{s} U_{21}(t,s)dtds \qquad (1)
\end{aligned}$$

This immediately gives monotonicity, i.e. $U_{21}(t,s) \geq 0$ almost everywhere. If not, then there exists some square $[\hat{f}, f]^2 \in [\underline{f}, \bar{f}]^2$ such that $U_{21}(t,s) < 0 \quad \forall (t,s) \in [\hat{f}, f]^2$, violating (1). Note that the inequality applies whether f is greater than or less than $\hat{f}$, because the double integral cancels the effect of integrating backwards. So I have shown that (LIC) and (M) are necessary conditions for (GIC).

To show sufficiency, note that (LIC) implies that (GIC) is characterized by (1), which is clearly satisfied when (M) holds. □

In this setting, where $U(\hat{f}|f) = m(\theta(\hat{f}))(\hat{f} - \delta f) - k\theta(\hat{f})$, we have the following characterization for GIC:

1. LIC:
$$0 = U_1(f|f) = (m'(\theta)(1-\delta)f - k)\,\theta'(f) + m(\theta) \qquad (2)$$

2. M:
$$0 \leq U_{12}(\hat{f}|f) = -\delta m'(\theta)\theta'(\hat{f}), \quad or \quad \theta'(f) \leq 0 \qquad (3)$$

The function $\theta(f)$ characterized by (2) is sensitive to the initial condition $\theta(\underline{f})$. Note that at $\underline{f}$, the coefficient on $\theta'(f)$ is zero at $\theta = \theta_{CI}(\underline{f})$. So if $\theta(\underline{f}) > \theta_{CI}(\underline{f})$, then $\theta'(f) > 0$, violating (M). If $\theta(\underline{f}) < \theta_{CI}(\underline{f})$, then $\theta'(f) < 0$, consistent with (M). If $\theta(\underline{f}) = \theta_{CI}(\underline{f})$, then an increasing or decreasing $\theta(f)$ is consistent with (2).

The above reasoning indicates that $\theta(\underline{f})$ must be less than or equal to $\theta_{CI}(\underline{f})$ in order to satisfy (M). I select the decreasing $\theta(f)$ with initial condition $\theta(\underline{f}) = \theta_{CI}(\underline{f})$, as it is the most liquid (highest profit) $\theta(f)$ which satisfies (LIC) and (M). This completes the proof of part 1 of the proposition, and part 2 follows directly from the buyer's free-entry condition.

It remains to show that this mechanism is decentralizable; i.e., that buyers cannot profit by offering an off equilibrium $p \notin P$. First note that $p^*(f)$ is continuous, with derivative $p^{*\prime}(f) = 1 - k\theta'(f)(m(\theta) - m'(\theta)\theta))/m(\theta)^2$. Because $\theta'(f) < 0$, and the concavity of m implies $m(\theta)/\theta > m'(\theta)$, I must have $p^{*\prime}(f) > 0$, so $P^* = [p^*(\underline{f}), p^*(\bar{f})] \subset \mathbb{R}_+$. Denote the lower and upper bounds of P^* by $\underline{p}$ and $\bar{p}$, respectively. It suffices to show that $p < \underline{p}$ and $p > \bar{p}$ are not profitable deviations for the buyer. Now suppose that a coaltion of buyers posts $p > \bar{p}$. Recall that $\theta(p, f) \equiv \inf\{\theta \geq 0 : m(\theta)(p - \delta f) \geq \Pi(f)\}$, and that buyers expect types $T(p) = \arg\inf_f \theta(p, f)$ (if any) and market tightness $\theta(p) = \inf_f \theta(p, f)$ for posting off-equilibrium p. Then if $p > \bar{p}$, then $\theta(p, f)$ satisfies $m(\theta(p, f))(p - \delta f) = \Pi(f) = m(\theta^*(f))(p^*(f) - \delta f)$. If so, then differentiating both sides with respect to f and applying the envelope condition shows that $\theta_2(p, f)$ has the same sign as $(p^*(f) - \delta f)/(p - \delta f) - 1$, so if $p > \bar{p}$, then $\theta(p, f)$ is minimized by $\bar{f}$, and $\theta(p) = \theta(p, \bar{f})$. Next, note that because $m(\theta(p))(p - \delta\bar{f}) = \Pi(\bar{f}) = m(\theta^*(\bar{f}))(p^*(\bar{f}) - \delta\bar{f})$, then $\theta(p) < \theta^*(\bar{f})$. Because $\theta^*(\bar{f}) < \theta_{CI}(\bar{f})$, and $m(\theta)(1 - \delta)\bar{f} - k\theta$ is increasing in θ for all $\theta < \theta_{CI}(\bar{f})$, it must be that $m(\theta(p)(1 - \delta)\bar{f} - k\theta(p) < m(\theta^*(\bar{f}))(1 - \delta)\bar{f} - k\theta^*(\bar{f})$. Multiply the zero profit function of a buyer who trades with $\bar{f}$ in equilibrium by $\theta^*(\bar{f})$ to get

$$
\begin{aligned}
0 &= m(\theta^*(\bar{f}))(\bar{f} - p^*(\bar{f})) - k\theta^*(\bar{f}) = m(\theta^*(\bar{f})(\bar{f} - \Pi(f)/m(\theta^*(\bar{f})) - \delta\bar{f}) - k\theta^*(\bar{f}) \\
&= -\Pi(\bar{f}) + m(\theta^*(\bar{f}))(1 - \delta)\bar{f} - k\theta^*(\bar{f}) > -\Pi(\bar{f}) + m(\theta(p))(1 - \delta)\bar{f} - k\theta(p) \\
&= -m(\theta(p))(p - \delta\bar{f}) + m(\theta(p))(1 - \delta)\bar{f} - k(\theta(p)) = m(\theta(p))(\bar{f}) - p) - k\theta(p)
\end{aligned}
$$

Divide both sides by $\theta(p)$ to get $0 > n(\theta(p))(\bar{f} - p) - k$, so $p > \bar{p}$ is not a profitable deviation.

On the other hand, if a coalition of buyers post $p < \underline{p}$, then because $\theta_2(p, f)$ has the same sign as $(p^*(f) - \delta f)/(p - \delta f) - 1$, $\theta_2(p, f)$ is minimized at $\underline{f}$, so buyers expect type $\underline{f}$, and therefore $\theta(p) = \theta(p, \underline{f})$. Note that the lowest type $\underline{f}$ receives the complete information liquidity $\theta^*(\underline{f}) = \theta_{CI}(\underline{f})$, which maximizes $m(\theta)(1-\delta)\underline{f} - k\theta$. Also note that $m(\theta(p))(p - \delta\underline{f}) = m(\theta^*(\underline{f}))(p^*(\underline{f}) - \delta\underline{f})$, so $\theta(p) > \theta^*(\underline{f})$. Therefore, $m(\theta^*(\underline{f}))(1-\delta)\underline{f} - k\theta^*(\underline{f}) > m(\theta(p))(1-\delta)\underline{f} - k\theta(p)$, so by the method above for $p > \bar{p}$, it must be that $p < \underline{p}$ is not a profitable deviation. □

Proof of Theorem 1. For ease of notation, I drop the * on all equilibrium functions, use $\theta(f)$ in place of $\Theta^*(f)$, and let $\underline{\theta} \equiv \Theta^*(\underline{f})$.

Part (i): The price function may be written as

$$P(\theta(f)) = f - \frac{k\theta(f)}{m(\theta(f))},$$

so I must show that the discount $k\theta/m(\theta)$ goes to zero as k goes to zero.

First consider the equilibrium for the lowest type $\underline{f}$. Recall that $\underline{\theta}$ solves the complete information FOC $m'(\underline{\theta})(1-\delta)\underline{f} = k$, so $\underline{\theta} \to \infty$ as $k \to 0$. Using the FOC, I can express the discount for $\underline{f}$ as

$$\frac{k\underline{\theta}}{m(\underline{\theta})} = (1-\delta)\underline{f}\frac{\underline{\theta}m'(\underline{\theta})}{m(\underline{\theta})}.$$

The following lemma guarantees that this discount goes to zero.

Lemma 2.

$$\lim_{\theta\to\infty} \theta m'(\theta) = 0$$

Proof. Observe that due to the concavity of m, for any $\hat{\theta} > 0$, if $\theta > \hat{\theta}$, then

$$m'(\theta) < \frac{m(\theta) - m(\hat{\theta})}{\theta - \hat{\theta}}.$$

If so, then for any $\hat{\theta} > 0$,

$$\lim_{\theta\to\infty} \theta m'(\theta) \leq \lim_{\theta\to\infty} \theta \left[\frac{m(\theta) - m(\hat{\theta})}{\theta - \hat{\theta}}\right] = \lim_{\theta\to\infty} \frac{m(\theta) - m(\hat{\theta})}{1 - \frac{\hat{\theta}}{\theta}} = 1 - m(\hat{\theta})$$

This holds for any $\hat{\theta}$, and since $\sup_{\hat{\theta}} m(\hat{\theta}) = 1$, I must have $\lim_{\theta\to\infty} \theta m'(\theta) = 0$. □

Now recall from monotonicity that $\theta(f) < \underline{\theta}$ for any $f > \underline{f}$, so as $k \to 0$,

$$0 < \frac{k\theta}{m(\theta)} < \frac{k\underline{\theta}}{m(\underline{\theta})} = (1-\delta)\underline{f}\frac{\underline{\theta} m'(\underline{\theta})}{m(\underline{\theta})} \to 0.$$

By the squeeze theorem, the discount $\frac{k\theta}{m(\theta)}$ for any f goes to zero, and therefore $P(\theta(f)) \to f$.

Part (ii): Let $\Theta(f,k)$ be the equilibrium θ for a given f and k. Let $h(\theta,k)$ be the inverse of $\Theta(f,k)$ so that $h(\Theta(f,k),k) = f$. If so, then $\Theta_1(h(\theta,k),k) = \frac{1}{h_1(\theta,k)}$. Then write (1) in terms of h:

$$\Big(m'(\theta)(1-\delta)h(\theta,k) - k\Big) = -m(\Theta)h_1(\theta,k),$$

Using integrating factors, I can solve for h explicitly:

$$h(\theta,k) = m(\theta)^{-(1-\delta)}\left(k\int m(\theta)^{-\delta}d\theta + C(k)\right),$$

where

$$\begin{aligned} C(k) &= \underline{f}m(\underline{\theta}(k))^{(1-\delta)} - k\int m(\theta)^{-\delta}d\theta\Big|_{\theta=\underline{\theta}(k)} \\ &= \underline{f}m(\underline{\theta})^{(1-\delta)} - (1-\delta)\underline{f}m'(\underline{\theta})\int m(\theta)^{-\delta}d\theta\Big|_{\theta=\underline{\theta}} \end{aligned}$$

Note that because $\lim_{\theta\to\infty} m(\theta)^{-\delta} = 1$, it must be that $\lim_{\underline{\theta}\to\infty} \int m(\theta)^{-\delta}d\theta\big|_{\theta=\underline{\theta}} = \infty$. Apply L'Hopital's Rule first to Lemma 2 and then to the second term of $C(k)$ to show that the second term converges to zero, which implies $C(k) \to \underline{f}$. This implies that as $k \to 0$, $h(\theta,k) \to m(\theta)^{-(1-\delta)}\underline{f} \equiv h(\theta)$. So then

$$P(\theta,k) = h(\theta,k) - \frac{k\theta}{m(\theta)} \to h(\theta) = \frac{\underline{f}}{[m(\theta)]^{1-\delta}}.$$

Part (iii): Note that $h(\theta)$ is invertible, so let $\Theta(f)$ be its inverse. Because $h(\Theta(f,k),k) = f$, it must be that $\Theta(f,k) \to \Theta(f)$. By the continuity of m,

$$\lim_{k \to 0} m(\Theta(f,k)) = m(\Theta(f)) = \left(\frac{\underline{f}}{h(\Theta(f))} \right)^{\frac{1}{1-\delta}} = \left(\frac{\underline{f}}{f} \right)^{\frac{1}{1-\delta}}.$$

□

Proof of Theorem 2. By Lemma 3 in the proof of the general multidimensional equilibrium (which applies to this special case), if there exists an incentive compatible mechanism which satisfies buyer free-entry, then there exists a Pareto dominant incentive compatible mechanism which satisfies buyer free-entry. I then use optimal control to characterize this mechanism, which turns out to be unique and fully separating, thereby proving that there exists a unique Pareto dominant fully separating mechanism. Finally, I show that buyers can't profit from deviating from the equilibrium set of prices p and quantities q, so the mechanism is decentralizable, and must therefore correspond to the Pareto optimal fully separating equilibrium.

The buyers' free-entry condition can be rearranged to obtain an expression for $P(\hat{f})$ in terms of $\theta(\hat{f})$ and $q(\hat{f})$:

$$P(\hat{f}) = \hat{f} - \frac{k\theta(\hat{f})}{m(\theta(\hat{f}))}.$$

I can then plug P into the seller's objective function (Definition 1) to obtain an expression for the type f seller's profit for reporting $\hat{f}$:

$$U(\hat{f}|f) = m(\theta(\hat{f}))q(\hat{f})(\hat{f} - \delta f) - k\theta(\hat{f}).$$

Applying Lemma 1, which was proved independent of the mechanism structure, I can characterize GIC in this context as

1. LIC:

$$0 = U_1(f|f) = (m'(\theta(f))q(f)(1-\delta)f - k)\,\theta'(f) +$$

$$m(\theta(f))(1-\delta)fq'(f) + m(\theta(f))q(f)$$

2. M:

$$0 \leq U_{12}(\hat{f}|f) = -\delta m'(\theta(\hat{f}))q(\hat{f})\theta'(\hat{f}) - \delta m(\theta(\hat{f}))q'(\hat{f}).$$

The problem is now to choose the Pareto optimal GIC equilibrium $(\theta(f), q(f))$. To do this, first observe that if two GIC equilibria i and j agree on $[\underline{f}, f]$, where $f < \bar{f}$, but disagree thereafter, then it must be that $\bar{U}_i(f) = \bar{U}_j(f)$ and $\bar{U}_i'(f) = -\delta m(\theta_i(f))q_i(f) = -\delta m(\theta_j(f))q_j(f) = \bar{U}_j'(f)$. Note however, that $\bar{U}_i''(f) = -\delta m'(\theta_i(f))q_i(f)\theta_i'(f) - \delta m(\theta_i(f))q_i'(f)$, and it is not clear that $\theta_i'(f) = \theta_j'(f)$ or that $q_i'(f) = q_j'(f)$, so it may be that $\bar{U}_i''(f) \neq \bar{U}_j''(f)$. A necessary condition for mechanism i to be Pareto optimal (which I refer to as *right dominance*) is for $\bar{U}_i''(f) \geq \bar{U}_j''(f)$ for all f. So the problem is to choose an incentive compatible equilibrium (θ, q) which maximizes $\bar{U}_i''(f)$ for all f. The solution given in the proposition has $\theta'(\underline{f}) = -\infty$, so I first solve the optimal control problem with the restriction $\theta' \geq -C$, where C is a positive constant, and then show that as $C \to \infty$, the solution converges to that in the proposition. I can now define the optimal control problem:

$$\max_{\theta', q'} -\delta m'(\theta)q\theta' - \delta m(\theta)q' \tag{Obj.}$$

s.t.

$$-\delta m'(\theta)q\theta' - \delta m(\theta)q' \geq 0 \tag{M}$$

$$\left(m'(\theta)q(1-\delta)f - k\right)\theta' + m(\theta)(1-\delta)fq' + m(\theta)q = 0 \tag{LIC}$$

$$\theta' \geq \begin{cases} 0 & \theta = 0 \\ -C & \theta > 0 \end{cases} \qquad q' \begin{cases} \leq 0 & q = 1 \\ \geq 0 & q = 0 \end{cases},$$

where the states are given by (θ, q), and the control variables are (θ', q'). It is straightforward to derive the solution for the following cases.

$$\theta' = \begin{cases} \frac{-m(\theta)}{m'(\theta)(1-\delta)f-k} & \theta > 0,\ q = 1, \text{ and } m'(\theta)(1-\delta)f - k > m(\theta)/C \\ 0 & \theta = 0 \text{ or } q = 0 \\ -C & \text{otherwise} \end{cases}$$

$$q' = \begin{cases} 0 & (\theta > 0,\ q = 1, \text{ and } m'(\theta)(1-\delta)f - k > m(\theta)/C) \text{ or } q = 0 \\ [-\infty, \infty] & \theta = 0 \\ \frac{-m(\theta)q+(m'(\theta)q(1-\delta)f-k)C}{m(\theta)(1-\delta)f} & \text{otherwise} \end{cases}$$

I set the initial condition for $\underline{f}$ equal to the complete information case, which maximizes the payoff to the lowest type. So the initial condition is $q(\underline{f}) = 1$ and $\theta(\underline{f})$ solves $m'(\theta)(1-\delta)\underline{f} = k$. Denote the solution to the q' differential equation above as $\tilde{q}(f, C)$, which can be solved for explicitly as $\tilde{q}(f, C) \equiv [m(\theta(\underline{f}) - Cf)]^{-1}\left(M_1 f^{-1/(1-\delta)} - kC\right)$, where M_1 is a constant chosen so that $\tilde{q}(\underline{f}) = 1$. The function $\tilde{q}(f, C)$ is U-shaped and has a positive vertical asymptote at $f = \theta(\underline{f})/C$. Denote by $f^*(C)$ the unique $f \in (\underline{f}, \theta(\underline{f})/C)$ such that $\tilde{q}(f, C) = 1$. In otherwords, the function $\tilde{q}(f, C)$ begins at 1, slopes downward and then upward in a U-shape, and then crosses back over $q = 1$ at $f^*(C)$ before asymptoting to $+\infty$ at $f = \theta(\underline{f})/C$.

Denote the solution to the differential equation for θ' above as $\tilde{\theta}(f, C)$, with initial condition $\tilde{\theta}(f^*(C), C) = \theta(\underline{f}) - Cf^*(C)$.

I can now define piecewise the Pareto optimal separating equilibrium, given the restriction that $\theta' \geq -C$.

$$q(f, C) = \begin{cases} \tilde{q}(f, C) & \underline{f} \leq f \leq f^*(C) \\ 1 & f^*(C) \leq f \leq \bar{f} \end{cases}$$

$$\theta(f, C) = \begin{cases} \theta(\underline{f}) - Cf & \underline{f} \leq f \leq f^*(C) \\ \tilde{\theta}(f, C) & f^*(C) \leq f \leq \bar{f} \end{cases}$$

It remains to be shown that this equilibrium converges to the one identified in the proposition as $C \to \infty$.

First consider $q(f, C)$. As explained above, there exists a unique $f^*(C)$ such that $f^* \in (\underline{f}, \theta(\underline{f})/C)$ and $\tilde{q}(f^*, C) = 1$. Because $\theta(\underline{f})/C \to 0$ as $C \to \infty$, it must be that $f^*(C) \to 0$ as well, implying that $\forall f \in [\underline{f}, \bar{f}]$, $q(f, C) \to 1 = Q^*(f)$.

Now consider $\theta(f, C)$. Because $f^*(C) \to 0$ as $C \to \infty$, the linear portion of $\theta(f, C)$ occupies a smaller and smaller interval $[\underline{f}, f^*(C)]$. However, the slope of that segment also becomes more and more negative, so it is not immediately clear if $\theta(f, C)$ converges to the convex curve identified in the proposition. I next show that the value of $\theta(f^*(C), C)$ at the piecewise boundary $f^*(C)$ converges to $\theta(\underline{f})$, the full information liquidity of the lowest type, which is sufficient to prove convergence to the equilibrium in the proposition.

Now denote $\hat{\theta} \equiv \theta(f^*(C), C)$ as the value of $\theta(f, C)$ where $\tilde{q}(f, C) = 1$ and $\theta(f, C)$ transitions from the linear segment to the convex curve defined by the differential equation. For convenience, denote $\underline{\theta} \equiv \theta(\underline{f})$ to be the initial condition, the full information liquidity to the lowest type. Because $\hat{\theta} = \theta(f^*(C), C) = \underline{\theta} - Cf^*$ and $\tilde{q}(f^*, C) = 1$, I can write $\tilde{q}((\underline{\theta} - \hat{\theta})/C, C)$ to characterize $\hat{\theta}$. Now substitute $x \equiv 1/C$ and rearrange to obtain the fixed point problem:

$$\hat{\theta} = H(\hat{\theta}, x) \equiv \underline{\theta} - \frac{\underline{f}}{x}\left[\left(\frac{m(\underline{\theta})x + k}{m(\hat{\theta})x + k}\right)^{1-\delta} - 1\right]$$

It can easily be shown that $H_{1,1}(\hat{\theta}, x) < 0$ for all $(\hat{\theta}, x) \in \mathbb{R}^2_+$, that $H_1(\underline{\theta}, x) = k/(m(\underline{\theta})x + k) < 1$ for $x > 0$, and that $\lim_{x\to 0} H_1(\underline{\theta}, x) = 1$. These features of H imply that for small enough $x > 0$, there exists exactly one fixed point $\hat{\theta} = H(\hat{\theta}, x)$ such that $\hat{\theta} < \underline{\theta}$, and that as $x \to 0$ it must be that $\hat{\theta} \to \underline{\theta}$. Therefore, as $C \to \infty$, I must have that $\theta(f^*(C), C) = \hat{\theta} \to \underline{\theta} = \theta_{CI}(\underline{f})$, and so for all $f \in [\underline{f}, \bar{f}]$, $\theta(f, C) \to \Theta^*(f)$, so the Pareto optimal incentive compatible mechanism has been found.

The optimal control selects a mechanism which satisfies right dominance of IC mechanisms satisfying free entry. This selected mechanism is IC, so there exists an IC mechanism. By the Lemma 3, there exists a Pareto optimal

IC mechanism satisfying buyer free entry; it must satisfy the right-dominance criterion, and this selected mechanism is the only mechanism which does. So this mechanism must be the unique Pareto optimal IC mechanism satisfying buyer free entry. The mechanism involves strictly decreasing liquidity, so it is fully separating; it is therefore the unique Pareto optimal fully separating mechanism.

I now check that buyers do not have a profitable off-equilibrium deviation. First observe the equilibrium prices p occupy the same space $[\underline{p}, \bar{p}]$ as in the one-signal case, but equilibrium q is equal to 1 everywhere, so $M^* = [\underline{p}, \bar{p}] \times 1$. First observe that $m(\theta(p,q,f))q(p - \delta f) = \Pi(f)$. Following the same reasoning as in the case with one signal, $\theta_3(p,q,f)$ has the same sign as $(p^*(f) - \delta f)/(p - \delta f) - 1$. So if buyers deviate with $(p,q) \in [\underline{p}, \bar{p}] \times (0,1)$, then $\theta(p,q,f)$ is minimized by the unique $f \in [\underline{f}, \bar{f}]$ that chooses p in equilibrium; i.e., $p^*(f) = p$. Therefore, $\theta(p,q) = \theta(p,q,p^{*-1}(p))$, and I have $m(\theta(p,q))q(p-\delta f) = \Pi(f) = m(\theta^*(f))(p^*(f)-f)$, where f is the type which chooses $p^*(f) = p$ in equilibrium (and therefore $(p - \delta f)$ and $(p^*(f) - \delta f)$ cancel). This gives $m(\theta(p,q)) = m(\theta^*(f))/q > m(\theta^*(f))$, because $q < 1$. Therefore, $\theta(p,q) > \theta^*(f)$, and because $n(\theta) = m(\theta)/\theta$ is strictly decreasing, I have $k = n(\theta^*(f))(f - p^*(f)) = n(\theta^*(f))(f - p) > n(\theta(p,q))q(f-p)$, so $(p,q) \in [\underline{p}, \bar{p}] \times (0,1)$ is not a profitable deviation. Next, suppose a coalition of buyers post a price $p > \bar{p}$ and $q \in [0,1]$. First recall that ;because $\theta_3(p,q,f)$ has the same sign as $(p^*(f)-\delta f)/(p-\delta f)-1$, for $p > \bar{p} \equiv p^*(\bar{f})$, $\theta(p,q,f)$ must be minimized by $\bar{f}$. So buyers expect type $\bar{f}$, and $\theta(p,q) = \theta(p,q,\bar{f}$ for $p > \bar{p}$. Next, observe that $\theta(p,1)$ satisfies $m(\theta(p,1))(p - \delta \bar{f}) = \Pi(f) = m(\theta^*(\bar{f}))(p^*(\bar{f}) - \delta \bar{f})$, so then $\theta(p,1) < \theta^*(\bar{f})$. Because $\theta^*(\bar{f})$ is less than the complete information θ and $\theta(p,1)$ is even lower, I must have $m(\theta^*(\bar{f})(\bar{f} - \delta \bar{f}) - k(\theta^*(\bar{f})) > m(\theta(p,1))(\bar{f} - \delta \bar{f}) - k\theta(p,1)$. So the zero-profit condition of buyers who trade with $\bar{f}$ in equilibrium gives

$$
\begin{aligned}
0 &= m(\theta^*(\bar{f}))(\bar{f} - p^*(\bar{f})) - k\theta^*(\bar{f}) = m(\theta^*(\bar{f}))(\bar{f} - \Pi(\bar{f})/m(\theta^*(\bar{f})) - \delta \bar{f}) \\
&= -\Pi(\bar{f}) + m(\theta^*(\bar{f}))(\bar{f} - \delta \bar{f}) - k\theta^*(\bar{f}) > -\Pi(\bar{f}) + m(\theta(p,1))(\bar{f} - \delta \bar{f}) - k\theta(p,1) \\
&= -m(\theta(p,1))(p - \delta \bar{f}) + m(\theta(p,1))(\bar{f} - \delta \bar{f}) - k\theta(p,1) = m(\theta(p,1))(\bar{f} - p) - k\theta(p,1)
\end{aligned}
$$

Dividing both sides by $\theta(p,1)$ gives $0 > n(\theta(p,1))(\bar{f}-p)-k$. Next, note that for $q \in [0,1]$, $m(\theta(p,q))q(p-\delta\bar{f}) = m(\theta(p,1))(p-\delta\bar{f})$, so $\theta(p,q) \geq \theta(p,1)$, and I have $n(\theta(p,q))q(\bar{f}-p)-k$, so $(p,q) \in (\bar{p},\infty)\times[0,1]$ is not a profitable deviation. Finally, suppose that a coalition posts $p < \underline{p}$ with $q \in [0,1]$. Again, because $\theta_3(p,q,f)$ has the same sign as $(p^*(f)-\delta f)/(p-\delta f)-1$, if (p,q) attracts any type, it must be $\underline{f}$, so $\theta(p,q) \equiv \theta(p,q,\underline{f})$. Recall that $\underline{f}$ receives the complete information liquidity, and because $m(\theta(p,q))q(p-\delta\underline{f}) = \Pi(f) = m(\theta^*(\underline{f}))(p^*(f)-\delta\underline{f})$ and therefore $\theta(p,q) > \theta^*(\underline{f}) = \theta_{CI}(\underline{f})$, it must be that $m(\theta^*(ulf))(\underline{f}-\delta\underline{f})-k\theta^*(\underline{f}) > m(\theta(p,q))(\underline{f}-\delta\underline{f})-k\theta(p,q)$. So applying the same reasoning as for $p > \bar{p}$, I have $0 > n(\theta(p,q))(\underline{f}-p)-k \geq n(\theta(p,q))q(\underline{f}-p-k)$, so $(p,q) \in [0,\underline{p})\times[0,1]$ is not a profitable deviation. □

Proof of Proposition 2 The only change from the previous problem is that the buyer free-entry condition is $n(\theta)q(E[\tilde{f}|p,q]-p) = kq$. The details of the proof are similar to that of Theorem 2, so I focus here on the optimal control problem for the case $\theta > 0$, $q = 1$, where the objective has been scaled by $1/\delta$. Suppose that $\theta > 0$ and $q = 1$, so that $-q' \geq 0$. Then the optimal control problem is

$$\max_{-\theta' \in \mathbb{R}, -q' \geq 0} m'(\theta)q(-\theta') + m(\theta)(-q') \tag{Obj.}$$

s.t.

$$\left(m'(\theta)(1-\delta)f-k\right)q(-\theta') + \left(m(\theta)(1-\delta)f-k\theta\right)(-q') = m(\theta)q \tag{LIC}$$

where the states are given by (θ,q), and the control variables are (θ',q'). Taking first order conditions and eliminating the Lagrange multiplier on the LIC constraint gives:

$$\frac{m'(\theta)q}{(m'(\theta)(1-\delta)f-k)q} = \frac{m(\theta)+\nu}{m(\theta)(1-\delta)f-k\theta},$$

where ν is the multiplier for the constraint $-q' \geq 0$. Rewrite the equation as

$$\frac{1}{(1-\delta)f-k/m'(\theta)} = \frac{1+\nu/m(\theta)}{(1-\delta)f-k\theta/m(\theta)}.$$

By the concavity of $m(\cdot)$, $m'(\theta) < m(\theta)/\theta$, so $\nu > 0$, and therefore $-q' = 0$, so retention shuts down. Plugging $-q' = 0$ and $q = 1$ into the LIC constraint gives the familiar ODE characterizing $\theta(f)$. Checking that off-equilibrium (p, q) are not profitable to buyers is similar to the proof for Theorem 2, and is left to the reader. $\square$

Proof of Lemma 1. First I show that global incentive compatibility implies the conditions in the lemma. Global incentive compatibility implies that

$$\begin{aligned}\Pi(\hat{s}|\hat{s}) &\geq \Pi(s|\hat{s}) \\ &= \Pi(s|s) + \Pi(s|\hat{s}) - \Pi(s|s) \\ &= \Pi(s|s) - m(\tilde{\theta}(s))\tilde{q}(s)(\hat{v} - v)\end{aligned}$$

Switching $\hat{s}$ and s and combining inequalities gives

$$-m(\tilde{\theta}(s))\tilde{q}(s)(\hat{v} - v) \leq \Pi(\hat{s}|\hat{s}) - \Pi(s|s) \leq -m(\tilde{\theta}(\hat{s}))\tilde{q}(\hat{s})(\hat{v} - v) \tag{4}$$

Part (i): Clearly, if $\hat{v} = v$, then $\Pi(\hat{s}|\hat{s}) = \Pi(s|s)$, regardless of $\hat{f}$ and f. Therefore, $\Pi((f, v)|(f, v))$ is constant in f, and is fully determined by v.

Part (ii): For shorthand, write $\Pi((f, v)|(f, v)) = \Pi(v)$, and rewrite (4) as

$$-m(\tilde{\theta}(s))\tilde{q}(s)(\hat{v} - v) \leq \Pi(\hat{v}) - \Pi(v) \leq -m(\tilde{\theta}(\hat{s}))\tilde{q}(\hat{s})(\hat{v} - v). \tag{5}$$

The right hand inequality indicates that if $\hat{v} > v$, then $\Pi(\hat{v}) \leq \Pi(v)$, so Π is decreasing in v. Next, for any $\epsilon > 0$, choose an arbitrary set of disjoint intervals (a_k, b_k) in $V = [\underline{v}, \bar{v}]$ such that $\sum_{k=1}^{N}(b_k - a_k) < \epsilon$ and set of arbitrary asset qualities $\{f_k\}$ so that $f_k \in \tilde{S}(a_k)$. Then

$$\begin{aligned}\sum_{k=1}^{N} |\Pi(b_k) - \Pi(a_k)| &= -\sum_{k=1}^{N}(\Pi(b_k) - \Pi(a_k)) \\ &\leq \sum_{k=1}^{N} m(\tilde{\theta}(f_k, a_k))\tilde{q}(f_k, a_k)(b_k - a_k) \\ &\leq \sum_{k=1}^{N}(b_k - a_k) < \epsilon,\end{aligned}$$

so Π is absolutely continuous, and is therefore differentiable almost everywhere. Now if $\Pi'(v)$ exists, then dividing (5) by $(\hat{v} - v)$ and letting $\hat{v}$ go to v from above and below gives (ENV).

Part (iii): This follows directly from (5).

I now show that the conditions in the lemma imply global incentive compatibility. Denote $m(\tilde{\theta}(f, v))\tilde{q}(f, v)$ by $H(f, v)$. Next, take $\{\hat{s}, s\} \equiv \{(\hat{f}, \hat{v}), (f, v)\} \subset \tilde{S}$, and let $h(\cdot)$ be any function over $[v, \hat{v}]$ with $h(v) = f$, $h(\hat{v}) = \hat{f}$, and such that the graph of h is contained in $\tilde{S}$. The lemma's third condition guarantees that $H(h(\cdot), \cdot)$ is decreasing, and the second condition indicates that where Π' exists, which is almost everywhere, then $-H(h(v), v) = \Pi'(v)$. I now have

$$\begin{aligned}\Pi(\hat{s}|\hat{s}) = \Pi(\hat{v}) &= \Pi(v) + \int_v^{\hat{v}} \Pi'(t)dt = \Pi(v) - \int_v^{\hat{v}} H(h(t), t)dt \\ &\geq \Pi(v) - \int_v^{\hat{v}} H(f, v)dt = \Pi(v) - m(\tilde{\theta}(s))\tilde{q}(s)(\hat{v} - v) \\ &= \Pi(s|\hat{s}),\end{aligned}$$

where the second equality follows from the absolute continuity of Π, and the inequality follows from the fact that $H(h(\cdot), \cdot)$ is decreasing. Therefore, global incentive compatibility holds. □

Proof of Proposition 3. I first prove that among the set of mechanisms which are IC and satisfy a condition necessary but not sufficient for separation, there exists a Pareto optimum. I then use optimal control to characterize this mechanism, which turns out to be unique and fully separating, thereby proving that there exists a unique Pareto dominant fully separating mechanism. Finally, I show that buyers can't profit from deviating from the equilibrium set of prices p and quantities q, so the mechanism is decentralizable, and must therefore correspond to the Pareto optimal fully separating equilibrium.

Because I am interested in fully separating equilibria, I restrict attention to IC mechanisms which satisfy

$$n(\theta(s'))q(s')(f' - p(s')) = k, \quad \text{(A)}$$

hereafter denoted IC-A mechanisms. That is, the expectation of the asset quality $E[\tilde{f}|f']$ is equal to the seller's report f'. This condition is necessary but not sufficient for separation, as I have not yet imposed $(p,q)(f) \neq (p,q)(f')$ (though if there were to exist an IC-A mechanism which was not separating, it would not correspond to any decentralizable equilibrium, as $(p,q)(f) = (p,q)(f')$ violates separation and (A) violates pooling).

Lemma 3. *If there exists an IC-A mechanism, then there exists a Pareto dominant IC-A mechanism.*

Proof. The proof is an adaptation of Lemma 2.2.2 in Viswanthan (1987). Let $\Pi(a_i(s')|s) \equiv m(\theta_i(s'))q_i(s')(p_i(s') - \delta f)$. Given $s \in S$, define the set $A(s) = \{a_i(s)|a_i \text{ is IC-A}\}$. Because $p(s) \in [\delta f, f]$, $q(s) \in [0,1]$, and condition (A) implies $\theta(s) \in [0, n^{-1}(k/(f - \delta f))]$, the set $A(s)$ is bounded, so the closure of $A(s)$ is compact. $\Pi(a|s)$ is continuous in a over the closure of $A(s)$, so by the extreme value theorem, $\Pi(a|s)$ attains its supremum over the closure of $A(s)$. Therefore, there exists an allocation $a(s)$ in the closure of $A(s)$ and a sequence $\{a_n(s)\}_{n=0}^{\infty} \subset A(s)$ such that as $n \to \infty$, $a_n(s) \to a(s)$ and $\Pi(a_n(s)|s) \to \Pi(a(s)|s) = \sup\{\Pi(a_i(s)|s) \mid a_i(s) \in A(s)\}$. Taking $a(s)$ over all $s \in S$ defines a mechanism over S, which Pareto dominates all IC-A mechanisms.

By the continuity of the left-hand side of (A), the mechanism a satisfies (A) for each $s \in S$. To show that a is IC, suppose not. Then there exists $s' \neq s$ such that $\Pi(a(s')|s) > \Pi(a(s)|s)$. Let $\epsilon = \Pi(a(s')|s) - \Pi(a(s)|s) > 0$. By the continuity of $\Pi(a|s')$ in a and the fact that $a(s')$ is in the closure of $A(s')$, there exists an IC mechanism a_i such that $|\Pi(a(s')|s) - \Pi(a_i(s')|s)| < \epsilon/2$. Therefore, $\epsilon = \Pi(a(s')|s) - \Pi(a(s)|s) < \Pi(a_i(s')|s) + \epsilon/2 - \Pi(a(s)|s) \leq \Pi(a_i(s)|s) + \epsilon/2 - \Pi(a(s)|s) \leq \Pi(a(s)|s) + \epsilon/2 - \Pi(a(s)|s) = \epsilon/2$, which is a contradiction. So a is a Pareto dominant IC-A mechanism. □

As an aside, the proof holds for any subset of IC-A mechanisms. So it establishes more generally that the set of IC-A mechanisms, a join-semilattice[1]

[1]Given IC-A mechanisms a_1 and a_2, for any $s \in S$, define $a(s) = a_i(s)$, where $i \in$

with Pareto dominance as its partial order, is complete, from which the statement of the lemma follows.

Because an IC-A mechanism a_i satisfies Lemma 1, write the equilibrium profit under some a_i as $\Pi(v)$. By (5), $\Pi(v)$ is convex, so the right and left derivatives of Π exist everywhere; I denote them by Π'_+ and Π'_-, respectively. A necessary condition of Pareto optimality (which I refer to as *right dominance*) is that if for all $v' \le v$, an IC-A mechanism a_i attains the same profit $\Pi(v')$ as a Pareto optimal mechanism a, then $\Pi'_+(v)$ under a must be greater than or equal to $\Pi'_+(v)$ under a_i. I now use optimal control to find a mechanism which maximizes Π'_+ subject to constraints implied by IC-A.

Given $v \in V$, $\Pi(v) > 0$, and $\Pi'_-(v) \in \mathbb{R}$, choose $\Pi'_+(v)$, $\theta(f,v)$ and $q(f,v)$ to solve the following program:

$$\max_{\Pi'_+ \in \mathbb{R}, \theta \ge 0, q \in [0,1]} \Pi'_+$$

s.t.

$$-\Pi'_-(v) \ge m(\theta(f,v))q(f,v) \ge -\Pi'_+(v) \ge 0 \qquad \forall f \in [\underline{f}(v), \bar{f}(v)] \tag{6}$$

$$\Pi(v) = m(\theta(f,v))q(f,v)(f-v) - k\theta(f,v) \qquad \forall f \in [\underline{f}(v), \bar{f}(v)] \tag{7}$$

$$\Pi(\cdot) \text{ is continuous and convex} \tag{8}$$

where (6) and (8) follow from (5), and (7) follows from substituting (A) into the definition of the seller's profit. Note that $\Pi'_+(v)$ is constant for all $f \in [\underline{f}(v), \bar{f}(v)]$, whereas θ and q may vary across f. I first prove two useful lemmas which characterize the constraint set for $f = \underline{f}(v)$, and then show that this leads to a unique solution which satisfies the constraints for all $f \in [\underline{f}(v), \bar{f}(v)]$.

$\arg\max_j U(a_j(s)|s)$; i.e., a is the join of a_1 and a_2. Suppose wlog that for $s \neq s'$, $a(s) = a_1(s)$ and $a(s') = a_2(s')$. Then a satisfies condition (A) and $U(a(s)|s) = U(a_1(s)|s) \ge U(a_2(s)|s) \ge U(a_2(s')|s) = U(a(s')|s)$, so a is also IC. The set of IC-A mechanisms contains the join of each pair, and is therefore a join-semilattice.

Lemma 4. *Given* $\Pi > 0$*, and* $v \in V$*, let* $\psi(\Pi, v)$ *be the set of* $(\theta, q) \in \mathbb{R}_+ \times [0,1]$ *that attain* Π *at* $(\underline{f}(v), v)$*:*

$$\psi(\Pi, v) \equiv \{(\theta, q) \in \mathbb{R}_+ \times [0,1] : \Pi = m(\theta)q(\underline{f}(v) - v) - k\theta\}.$$

Let $\underline{\theta}(\Pi, v)$ *be the lowest* θ *that attains* Π *when* $q = 1$*:*

$$\underline{\theta}(\Pi, v) \equiv \inf\{\theta \geq 0 : \Pi = m(\theta)(\underline{f}(v) - v) - k\theta\}.$$

Then

(i) $\psi(\Pi, v)$ *is nonempty iff* $\Pi \leq \Pi_{CI}(\underline{f}(v), v)$*.*

(ii) $m(\theta)q$ *is minimized over* $\psi(\Pi, v)$ *by* $(\theta, q) = (\underline{\theta}(\Pi, v), 1)$*.*

Proof. Part (i): The function $m(\theta)q(\underline{f}(v) - v) - k\theta$ is continuous, unbounded below, and attains its maximum at $\Pi_{CI}(\underline{f}(v), v)$. Therefore, there exist $(\theta, q) \in \mathbb{R}_+ \times [0,1]$ which set $m(\theta)q(\underline{f}(v) - v) - k\theta$ equal to Π iff Π is in the range of $m(\theta)q(\underline{f}(v) - v) - k\theta$, which is $(-\infty, \Pi_{CI}(\underline{f}(v), v)]$.

Part (ii): The program is to minimize $m(\theta)q$ over $\psi(\Pi, v)$. If I solve for q in the definition of ψ, I may rewrite the program with q eliminated as follows:

$$\min_{\theta \geq 0} \frac{\Pi + k\theta}{\underline{f}(v) - v}$$

s.t.

$$\frac{\Pi + k\theta}{m(\theta)(\underline{f}(v) - v)} \in [0,1] \tag{9}$$

The objective function is increasing in θ, and the left-hand side of (9) goes to ∞ as $\theta \to 0$. Therefore, the program is solved when the left-hand side of (9) (i.e., q) is equal to 1, and θ equals $\underline{\theta}(\Pi, v)$. □

Lemma 5. *For all* $v \in V$*,* $-\Pi'_+(v)$ *is bounded below by* $m(\underline{\theta}(\Pi, v))$*.*

Proof. Suppose there exists a $v \in V$ such that $-\Pi'_+(v) < m(\underline{\theta}(\Pi, v))$. Let $\epsilon = m(\underline{\theta}(\Pi, v)) - (-\Pi'_+(v))$. By the continuity of $\Pi(\cdot)$, $\underline{\theta}(\cdot, \cdot)$, and $m(\cdot)$, there exists a $\epsilon' > 0$ such that for all $\hat{v} \in (v, v + \epsilon')$, $|m(\underline{\theta}(\Pi(v), v)) - m(\underline{\theta}(\Pi(\hat{v}), \hat{v}))| < \epsilon$.

Also, by the convexity of Π, I know that $-\Pi'_+(v) \geq -\Pi'_-(\hat{v})$. Combining these two facts with (6) at $\hat{v}$ yields:

$$m(\underline{\theta}(\Pi(\hat{v}), \hat{v})) > m(\underline{\theta}(\Pi(v), v)) - \epsilon = -\Pi'_+(v) \geq -\Pi'_-(\hat{v}) \geq m(\theta(\hat{v}))q(\hat{v}).$$

By Part (ii) of the prior lemma, this is a contradiction. □

The lemma says that for any $v \in V$, there is no IC mechanism in which the right-derivative of the profit function exceeds $-m(\underline{\theta}(\Pi(v), v))$. Furthermore, Part (i) of Lemma 4 implies that there is no IC mechanism in which the initial value $\Pi(\underline{v})$ exceeds the complete information profit $\Pi_{CI}(\underline{f}(\underline{v}, \underline{v}))$. Therefore, if there exists a mechanism that satisfies (6) - (8), with $\Pi'_+(v) = -m(\underline{\theta}(\Pi(v), v))$ at every $v \in V$, and initial value $\Pi(\underline{v}) = \Pi_{CI}(\underline{f}(\underline{v}, \underline{v}))$, it must solve the optimal control problem. I now construct such a mechanism and show it is unique.

If $\Pi'_+(v) = -m(\underline{\theta}(\Pi(v), v))$ for all $v \in V$, then by the continuity of $m(\underline{\theta}(\Pi(\cdot), \cdot))$, $\Pi'_+(v)$ must be continuous in V. This, together with the continuity of Π, implies that Π is differentiable everywhere, so for all $v \in V$, $\Pi'_-(v) = \Pi'_+(v)$, and (6) must hold with equality. Now for all $f \in [\underline{f}(v), \bar{f}(v)]$, equations (6) and (7) uniquely pin down $\theta(f, v)$ and $q(f, v)$ as follows:

$$\theta(f, v) = \frac{1}{k}\Big[-\Pi(v) - \Pi'(v)(f - v)\Big], \qquad q(f, v) = \frac{-\Pi'(v)}{m(\theta(f, v))},$$

where $-\Pi'(v) = m(\underline{\theta}(\Pi(v), v))$. Recalling the definition of $\underline{\theta}(\Pi(v), v)$, write

$$\underline{\theta}(\Pi(v), v) = \frac{1}{k}\Big[-\Pi(v) - m(\underline{\theta}(\Pi(v), v))(\underline{f}(v) - v)\Big],$$

where $\underline{\theta}(\Pi(v), v)$ is the lowest fixed point of the right-hand side. Apply $m(\cdot)$ to both sides and substitute $-\Pi'(v) = m(\underline{\theta}(\Pi(v), v))$ to get the expression in the proposition:

$$-\Pi'(v) = m\left(\frac{1}{k}\Big[-\Pi(v) - \Pi'(v)(\underline{f}(v) - v)\Big]\right) \quad \text{with} \quad \Pi(\underline{v}) = \Pi_{CI}(\underline{f}(v), v),$$

where $\Pi'(v)$ is the lowest fixed point of the right-hand side of the ODE. The solution $\Pi(\cdot)$ to this initial value problem is unique, strictly convex, and of course continuous, so the proposed mechanism satisfies (8), and by construction it satisfies (6) and (7). It remains to be shown that θ and q take on feasible values.

Substituting $-\Pi'(v) = m(\underline{\theta}(\Pi(v), v))$ in the expression for $\theta(f, v)$ and noting $f \geq \underline{f}(v)$, I have

$$\theta(f, v) \geq \frac{1}{k}\Big[-\Pi(v) - m(\underline{\theta}(\Pi(v), v))(\underline{f}(v) - v)\Big] = \underline{\theta}(\Pi(v), v) > 0,$$

where the last inequality is strict as long as $\Pi > 0$, which is true everywhere in the unique solution to the ODE. This expression implies

$$q(f, v) = \frac{-\Pi'(v)}{m(\theta(f, v))} = \frac{m(\underline{\theta}(\Pi(v), v))}{m(\theta(f, v))} \in [0, 1],$$

so both $\theta(f, v)$ and $q(f, v)$ are in their feasible sets, and I have found the solution to the optimal control problem. Note that although the constraints of the optimal control problem were necessary but not sufficient for IC, it can be easily shown that this solution satisfies the assumptions in Lemma 1, so it is IC.

The optimal control selects a mechanism which satisfies right dominance of IC-A mechanisms. This selected mechanism is IC-A, so there exists an IC-A mechanism. By the Lemma 3, there exists a Pareto optimal IC-A mechanism; it must satisfy the right-dominance criterion, and this selected mechanism is the only mechanism which does. So this mechanism must be the unique Pareto optimal IC-A mechanism. Condition (A) implies a price $p(f, v)$ which is strictly increasing in v, and $q(f, v)$ is strictly decreasing in f, so the mechanism is fully separating, and because separation implies IC-A, it is therefore the unique Pareto optimal fully separating mechanism.

I now show that the mechanism is decentralizable. The form for p^* and q^* in the Proposition follow directly from the form of the mechanism characterized above. Because the mechanism is IC, it is clear that the strategies p^* and q^* are optimal to the seller, and by construction the mechanism satisfies the buyer's zero-profit condition. It remains to check that buyers cannot profit from offering a $(p, q) \notin M^*$.

Observe that because $p^*(f,v)$ is independent of f, continuous, and strictly increasing in v, the set of prices posted in equilibrium is the closed interval $[\underline{p},\bar{p}]$, where I let $\underline{p} \equiv p^*(\underline{v})$ and $\bar{p} \equiv p^*(\bar{v})$. Next, recall that in the Pareto optimal mechanism, $q(\underline{f}(v),v) = 1$ for all $v \in V$, and observe that $q^*(f,v)$ is continuous and strictly decreasing in f. Therefore, for any $p \in [\underline{p},\bar{p}]$, the set of quantities q posted is the closed interval $[\underline{q}(p),1]$, where $\underline{q}(p) \equiv q^*(\bar{f}(v),v)$ and $p = p^*(v)$. Because M^* takes this form, it suffices to show that $p > \bar{p}$ and $p < \underline{p}$ are not profitable deviations, and that if p is an element of $[\underline{p},\bar{p}]$, then $q < \underline{q}(p)$ is not a profitable deviation.

Suppose that a coalition of buyers consider posting a (p,q) pair where $p \in [\underline{p},\bar{p}]$, but $q < \underline{q}(p)$. First define $\theta((p,q),s) \equiv \inf\{\tilde{\theta} > 0 : m(\theta)q(p-v) \geq \Pi(v)\}$, which is the lowest acceptable market tightness for type s in equilibrium. Recall that when buyers post off-equilibrium (p,q) pairs, they expect the type that will accept the lowest probability of trade. Because $\theta((p,q),s)$ depends strictly on v, buyers' off-equilibrium beliefs are a distribution over $T(p,q) = \arg\inf_{s\in\tilde{S}} \theta((p,q,),s)$. Noting that $m(\theta(p,q,v))q(p-v) = \Pi(v)$, differentiating by v and substituting the expression for $p^*(v)$ shows that $\theta_3(p,q,v)$ is proportional to $(p^*(v)-v)/(p-v)-1$, so $\theta(p,q,v)$ is clearly minimized over V by $p^{*-1}(p)$, the unique type v which selects p in equilibrium. So buyers off-equilibrium beliefs about asset quality f are a distribution over $\tilde{S}(v)$. Also recall that for off-equilibrium (p,q), $\theta(p,q) \equiv \inf_{s\in\tilde{S}} \theta((p,q,s))$. So if v is the minimizing value, then $m(\theta(p,q))q(p-v) = \Pi(v) = m(\theta(f,v))q(f,v)(p^*(v)-v)$, where v is the type which chooses $p^*(v) = p$ in equilibrium (and therefore $(p-v)$ and $(p^*(v)-v)$ cancel), and $f \in \tilde{S}(v)$. This gives $m(\theta(p,q)) = m(\theta(f,v))q(f,v)/q > m(\theta(f,v))$, because $q < \underline{q}(p) \leq q(f,v)$. Therefore, $\theta(p,q) > \theta(f,v)$, and because $n(\theta) = m(\theta)/\theta$ is strictly decreasing, for any $f \in \tilde{S}(v)$, I have $k = n(\theta(f,v))q(f,v)(f-p) = n(\theta(\bar{f}(v),v))q(\bar{f}(v),v)(\bar{f}(v) - p) > n(\theta(p,q))q(E[\tilde{f}|p,q]-p)$, where I have used the fact that the buyer's expectation over $\tilde{S}(v)$ cannot exceed $\max\{\tilde{S}(v)\} = \bar{f}(v)$. So regardless of the buyer's off-equilibrium belief distribution over asset qualities $f \in \tilde{S}(v)$, $(p,q) \in [\underline{p},\bar{p}] \times [0,\underline{q}(p))$ is not a profitable deviation.

Next, suppose that a coalition posts $p > \bar{p}$, with any $q \in [0,1]$. As explained above, $\theta_3(p,q,v)$ has the same sign as $(p^*(v)-v)/(p-v)-1$, so $\theta(p,q,v)$ must be minimized by $v = \bar{v}$. Note that $\tilde{S}(v) = \{\bar{f}\}$, the highest possible asset quality in $\tilde{S}$, not just $\tilde{S}(v)$, so buyers expect type $(\bar{f},\bar{v})$. Next, observe that $\theta(p,1)$ satisfies $m(\theta(p,1))(p-\bar{v}) = \Pi(\bar{v}) = m(\theta(\bar{f},\bar{v}))(p^*(\bar{v})-\bar{v})$, where I have used $q(\bar{f},\bar{v}) = 1$; so then, $\theta(p,1) < \theta(\bar{f},\bar{v})$. Because $\theta(\bar{f},\bar{v})$ is less than the complete information θ, and $\theta(p,1)$ is even lower, I must have $m(\theta(\bar{f},\bar{v}))(\bar{f}-\bar{v}) - k\theta(\bar{f},\bar{v}) > m(\theta(p,1))(\bar{f}-\bar{v}) - k\theta(p,1)$ So the zero-profit condition of buyers who trade with $(\bar{f},\bar{v})$ in equilibrium gives

$$\begin{aligned}0 &= m(\theta(\bar{f},\bar{v}))(\bar{f}-p^*(\bar{v})) - k\theta(\bar{f},\bar{v}) = m(\theta(\bar{f},\bar{v}))(\bar{f}-\Pi(\bar{v})/m(\theta(\bar{f},\bar{v}))-\bar{v}) - k\theta(\bar{f},\bar{v})\\ &= -\Pi(\bar{v}) + m(\theta(\bar{f},\bar{v}))(\bar{f}-\bar{v}) - k\theta(\bar{f},\bar{v}) > -\Pi(\bar{v}) + m(\theta(p,1))(\bar{f}-\bar{v}) - k\theta(p,1)\\ &= -m(\theta(p,1))(p-v) + m(\theta(p,1))(\bar{f}-\bar{v}) - k\theta(p,1) = m(\theta(p,1))(\bar{f}-p) - k\theta(p,1).\end{aligned}$$

Dividing both sides by $\theta(p,1)$ gives $0 > n(\theta(p,1))(\bar{f}-p)-k$. Next, note that for $q \in [0,1]$, $m(\theta(p,q))q(p-\bar{v}) = m(\theta(p,1))(p-\bar{v})$, so $\theta(p,q) \geq \theta(p,1)$, and I have $0 > n(\theta(p,q))q(\bar{f}-p)-k$, so $(p,q) \in (\bar{p},\infty)\times[0,1]$ is not a profitable deviation.

Suppose that a coalition posts $p < \underline{p}$, with $q \in [0,1]$. Because $\theta_3(p,q,v)$ has the same sign as $(p^*(v)-v)/(p-v)-1$, if (p,q) attracts any type, it must be $\underline{v}$ and therefore $(\underline{f},\underline{v})$. If so, then $q(p-\underline{v}) > m(\theta(p,q))q(p-\underline{v}) = \Pi(\underline{v}) = m(\theta(\underline{f},\underline{v}))\cdot 1\cdot(p^*(\underline{v})-\underline{v})$. Therefore, $q > m(\theta(\underline{f},\underline{v}))(p^*(\underline{v})-\underline{v})/(p-\underline{v}) > m(\theta(\underline{f},\underline{v}))$. Also require that $p-v > m(\theta(\underline{f},\underline{v}))(p^*(v)-v)$. Otherwise, no types are attracted, $\theta(p,q) = \infty$, and the buyer's profit is $-k$. With these restrictions on q and p, recall that $(\underline{f},\underline{v})$ receives the complete information allocation, so $m(\theta(\underline{f},\underline{v})(\underline{f}-\underline{v}) - k\theta(\underline{f},\underline{v}) \geq m(\theta(p,q))q(\underline{f}-\underline{v}) - k\theta(p,q)$. So the zero-profit condition of buyers who trade with $(\underline{f},\underline{v})$ in equilibrium gives

$$\begin{aligned}0 &= m(\theta(\underline{f},\underline{v}))(\underline{f}-p^*(\underline{v})) - k\theta(\underline{f},\underline{v}) = m(\theta(\underline{f},\underline{v}))(\underline{f}-\Pi(\underline{v})/m(\theta(\underline{f},\underline{v}))-\underline{v}) - k\theta(\underline{f},\underline{v})\\ &= -\Pi(\underline{v}) + m(\theta(\underline{f},\underline{v}))(\underline{f}-\underline{v}) - k\theta(\underline{f},\underline{v}) \geq -\Pi(\underline{v}) + m(\theta(p,q))q(\underline{f}-\underline{v}) - k\theta(p,q)\\ &= -m(\theta(p,q))q(p-v) + m(\theta(p,q))q(\underline{f}-\underline{v}) - k\theta(p,q) = m(\theta(p,q))q(\underline{f}-p) - k\theta(p,q).\end{aligned}$$

Dividing both sides by $\theta(p,q)$ gives $0 \geq n(\theta(p,q))q(\underline{f}-p)-k$, so $(p,q) \in [0,\underline{p}] \times [0,1]$ is not a profitable deviation, and the proof is complete. $\square$

Proof of Corollary 1 These follow directly from differentiating the expressions for p^*, $m(\theta^*)q^*$, and $\tilde{\theta}(f,v)$ in Proposition 3 and the strict convexity of the profit function $\Pi(v)$. $\square$

Proof of Corollary 2 These follow directly from differentiating the expressions for $\tilde{\theta}$ and $\tilde{q}$ in Proposition 3. $\square$

Proof of Corollary 3 This is stated in the form of the equilibrium.

Proof of Corollary 4 *Part (i):* Follows from Corollary 3. *Part (ii):* Recall from the proof of Proposition 3 that $-\Pi'(v) = m(\theta(\underline{f},v))q(\underline{f},v) = m(\theta(\underline{f},v))$, where the second equality follows from Corollary 3. Differentiate both sides with respect to v to obtain

$$-\Pi''(v) = m'(\theta(\underline{f},v))\theta_v(\underline{f},v) \tag{10}$$

$$= -m'(\theta(\underline{f},v))\Pi''(v)(\underline{f}-v)/k, \tag{11}$$

where the second equality follows from differentiating the expression for $\theta(f,v)$ in Proposition 3. The strict convexity of Π implies that $\Pi''(v) > 0$ and may be cancelled, yielding $m'(\theta(\underline{f},v))(\underline{f}-v) = k$, which characterizes the complete information liquidity $\theta(\underline{f},v)$. $\square$

Proof of Theorem 3. *Part (i):* Denote the liquidity and fraction sold over domain $\tilde{S}$ by $\tilde{\theta}$ and $\tilde{q}$, and over domain S by θ and q. Then $\theta(f,\delta) = \tilde{\theta}(f,\delta f)$ and $q(f,\delta) = \tilde{q}(f,\delta f)$. This gives

$$\begin{aligned}\theta_1 &= \tilde{\theta}_1 + \tilde{\theta}_2\delta \\ &= -\frac{1}{k}[\Pi'(v) + \delta\Pi''(v)(f-v)].\end{aligned}$$

Also note that differentiating (2) with respect to v gives

$$\left[m'(\underline{\theta}(v))(\underline{f}(v)-v)-k\right]\Pi''(v) = -\Pi'(v)m'(\underline{\theta}(v))\underline{f}'(v), \tag{12}$$

where I denote $\underline{\theta}(v) \equiv \tilde{\theta}(\underline{f}(v),v)$ as the liquidity of the lowest quality asset f for a given private valuation v, and therefore the argument of m' in (12) and of m in (2) is $\underline{\theta}(v)$. Use (12) to solve for $\Pi''(v)$, note that in $\bar{S}$, $\underline{f}(v) = v/\bar{\delta}$, and rearrange the above equation to get

$$\theta_1 = \frac{-\Pi'(v)}{k[m'(\underline{\theta}(v))(1-\bar{\delta})v/\bar{\delta} - k]}[-(\bar{\delta} - \delta)m'(\underline{\theta}(v))v/\bar{\delta} - k] < 0.$$

Parts (ii) and (iii): Consider $\theta_1(\underline{f}, \delta)$, which is the market tightness partial at a point on the left border of S. Recall that $k\theta(f, \delta) = -\Pi(\delta f) - \Pi'(\delta f)(f - \delta f)$. Differentiating with respect to f gives

$$k\theta_1(f, \delta) = -\Pi'(\delta f) - \Pi''(\delta f)(1-\delta)\delta f \tag{13}$$

For a given v, denote the liquidity associated with the lowest type $\underline{f}(v)$ as $\underline{\theta}(v) \equiv \tilde{\theta}(\underline{f}(v), v)$. In the low value region $\underline{S} = \{(f, \delta) : \delta f < \bar{\delta}\underline{f}\}$, $\underline{f}(v) = \underline{f}$, so $\underline{\theta}(v) = \tilde{\theta}(\underline{f}, v)$. Then I can compute $\Pi''(v)$ by noting that $\Pi'(v) = -m(\underline{\theta}(v))$. Recalling that in $\underline{S}$, $m'(\underline{\theta}(v))(\underline{f} - v) = k$, I have $\Pi''(v) = -m'(\underline{\theta}(v))\underline{\theta}'(v) = -\frac{[m'(\underline{\theta}(v))]^2}{m''(\underline{\theta}(v))(\underline{f}-v)}$. Returning to (13), I have

$$k\theta_1(f, \delta) = m(\underline{\theta}(v)) + \delta\frac{[m'(\underline{\theta}(v))]^2}{m''(\underline{\theta}(v))}\frac{(f-v)}{(\underline{f}-v)}.$$

Dividing both sides by $m(\underline{\theta}(v))$ and incorporating the assumed form of $m(\theta) = (1+\theta^{-r})^{-1/r}$, I have that $\theta_1(\underline{f}, \delta) > 0$ if and only if $(1+r)\underline{\theta}(v)^r > \delta(f-v)/(\underline{f} - v)$. Use $m'(\underline{\theta}(v))(\underline{f} - v) = k$ to solve for $\underline{\theta}(v)$, plug in the previous inequality, and rearrange to get that $\theta_1(\underline{f}, \delta) > 0$ if and only if

$$\left(\frac{\underline{f}(1-\delta)}{k}\right)^{\frac{r}{1+r}} > \frac{\delta}{1+r}\frac{f-v}{\underline{f}-v} + 1. \tag{14}$$

Assumption 1 guarantees that $\underline{f}(1-\delta)/k > 1$, so as $r \to \infty$, the limit of the left hand side of (14) is strictly greater than the limit of the righthand side, and therefore $\theta_1(f, \delta) > 0$ for high enough r. On the other hand, as $r \to 0$, the left hand side goes to 1, and the right hand side goes to a value strictly greater than 1, so the inequality reverses, and $\theta_1(f, \delta) < 0$. □

Proof of Proposition 4 The first part of the proof is nearly identical to the proof of Proposition 3, but with $(f - v)$ in (7) replaced with $(E[\tilde{f}|\tilde{\delta}\tilde{f} =$

$v] - v)$. Making this substitution, it is easy to show that the equilibrium identified in Proposition 4 corresponds to the Pareto optimal partial pooling mechanism.

To show that buyers are not motivated to deviate from the equilibrium set M^* of prices p and quantities q, first note that this set takes the form $M^* = [\underline{p}, \bar{p}] \times 1 \subset \mathbb{R}^2_+$. The proof that (p, q) with $p \notin [\underline{p}, \bar{p}]$ is not a profitable deviation is the same as in the proof of Proposition 3, so now consider a deviation (p, q), with $p \in [\underline{p}, \bar{p}]$ and $q < 1$. As discussed in the proof of Proposition 3, the buyer's beliefs must be distributed over the types $(f, v) \in \tilde{S}$ where v is the unique v for which $p^*(v) = p$. The literature do not restrict beliefs beyond that, so suppose that (in accordance with Guerrieri and Shimer (2013)), the buyer's belief corresponds to the actual distribution of sellers with private value v, so his expected asset quality is $E[\tilde{f}|\tilde{\delta}\tilde{f} = v]$. Recall as in the proof of Proposition 3 that $m(\theta(p, q))q(p - v) = m(\theta(v))(p - v)$, so $\theta(p, q) > \theta(v)$. If so, then $k = n(\theta(v))(E[\tilde{f}|\tilde{\delta}\tilde{f} = v] - p) > n(\theta(p, q))q(E[\tilde{f}|\tilde{\delta}\tilde{f} = v] - p)$, so (p, q) is not a profitable deviation, and the Proposition is proved. □

Proof of Proposition 5 Recall that under full separation Σ, the profit of any seller who privately values his asset as $v \equiv \delta f$ may be characterized by the following ODE:

$$\Pi'(v) = -m\left(-\frac{1}{k}\left[\Pi(v) + \Pi'(v)(\underline{f}(v) - v)\right]\right), \qquad \Pi(\underline{v}) = \Pi_{CI}(\underline{f}(\underline{v}), \underline{v}), \tag{15}$$

where $\underline{f}(v)$ is the lowest quality asset among sellers with common private value v, and $\Pi_{CI}(f, v)$ is the complete information profit of seller (f, v). The optimal control argument which generates the ODE may be similarly applied to the case of partial pooling Φ, yielding the following characterization:

$$\Pi'(v) = -m\left(-\frac{1}{k}\left[\Pi(v) + \Pi'(v)(E[\tilde{f}|\tilde{f}\tilde{\delta} = v] - v)\right]\right), \qquad \Pi(\underline{v}) = \Pi_{CI}(\underline{f}(\underline{v}), \underline{v}), \tag{16}$$

Clearly, both equilibria have the same initial condition $\Pi(\underline{v}) = \Pi_{CI}(\underline{f}(\underline{v}), \underline{v})$, and the only difference between (15) and (16) is that $E[\tilde{f}|\tilde{f}\tilde{\delta} = v]$ has been substituted for $\underline{f}(v)$. As long as some sellers of value v have assets better than $\underline{f}(v)$, it must be that $E[\tilde{f}|\tilde{f}\tilde{\delta} = v] > \underline{f}(v)$.

Now suppose that the two equilibria have the same profit $\Pi(v)$ for some v, and consider how $\Pi'(v)$ differs under the two equilibria. Let g be a placeholder for either expression $E[\tilde{f}|\tilde{f}\tilde{\delta} = v]$ or $\underline{f}(v)$, and let α be a placeholder for $\Pi'(v)$ and write

$$\alpha = -m\left(-\frac{1}{k}\left[\Pi(v) + \alpha(g - v)\right]\right). \tag{17}$$

Now fix v and $\Pi(v)$, and consider how α changes as g increases from $\underline{f}(v)$ to $E[\tilde{f}|\tilde{f}\tilde{\delta} = v]$. Differentiate both sides of (17) with respect to g, and solve for $\alpha'(g)$ to obtain

$$\alpha'(g) = \frac{m'(...)(-\alpha)}{m'(...)(g - v) - k} > 0, \tag{18}$$

where the inequality is due to the fact that $\alpha = \Pi'(v)$ is negative and the denominator $m'(...)(g - v) - k$ is positive. Therefore, $\Pi'(v)$ is higher under partial pooling where $g = E[\tilde{f}|\tilde{f}\tilde{\delta} = v] > \underline{f}(v)$ than under full separation where $g = \underline{f}(v)$, so wherever $\Pi_\Phi(v)$ crosses $\Pi_\Sigma(v)$, it must be that $\Pi'_\Phi(v) > \Pi'_\Sigma(v)$.

Finally, suppose there exists a v at which the profit function $\Pi_\Phi(v)$ under pooling is less than or equal to that under full separation $\Pi_\Sigma(v)$. Then because the two equilibrium profit functions are equal at the initial condition $\Pi(\underline{v}) = \Pi_{CI}(\underline{f}, \underline{v})$, the pooling profit $\Pi_\Phi(v)$ must cross the separating profit $\Pi_\Sigma(v)$ from above, which contradicts $\Pi'_\Phi(v) > \Pi'_\Sigma(v)$. Therefore, for all $v > \underline{v}$, $\Pi_\Sigma(v) < \Pi_\Phi(v)$, and Part (i) is proved.

Proof of Proposition 6 The proof of the fully separating equilibrium shows that Σ is robust to any belief satisfying (R), so therefore $\Gamma(\Sigma) \supseteq \Gamma_0$. By definition, any belief in $\Gamma(\Sigma)$ must satisfy (R), so $\Gamma(\Sigma) \subseteq \Gamma_0$. Combining these two relations, $\Gamma(\Sigma) = \Gamma_0$.

By definition, $\Gamma(\Phi) \subseteq \Gamma_0 = \Gamma(\Sigma)$, so what remains to be shown is that $\Gamma(\Phi)$ is a strict subset of $\Gamma(\Sigma)$. To do this, let $\hat{T}(p, q) \equiv \{(\bar{f}(v), v) : (f, v) \in T(p, q)\}$. I propose that any off-equilibrium beliefs with support restricted to $\hat{T}(p, q)$

support full separation Σ but break partial pooling Φ. The proof of Proposition 3 shows that if p is in the set of equilibrium prices, then $T(p,q) = \{(f,v) \in \tilde{S} : p^*(v) = p\}$. This indicates that as defined, $\hat{T}(p,q) \subset T(p,q)$. Therefore, beliefs which are restricted to have support no larger than $\hat{T}(p,q)$ satisfy restriction (R) and are therefore in $\Gamma(\Sigma)$. However, such beliefs are not in $\Gamma(\Phi)$. To see this, note that under partial pooling Φ, (16) indicates that all sellers choose $q = 1$, so $q < 1$ is off equilibrium. Suppose that buyers post (p,q), where p is in the equilibrium set of prices, but $q < 1$ and therefore off-equilibrium. Then $m(\theta(p,q))q(p-v) = \Pi(v) = m(\theta(p,1))(p-v)$, where $p^*(v) = p$. If so, then off-equilibrium $\theta(p,q)$ satisfies $m(\theta(p,q)) = m(\theta(p,1))/q$, and is therefore continuous in q. So letting $\epsilon = \bar{f}(v) - E[\tilde{f}|v]$, let $\epsilon' = n(\theta(p,1))\epsilon/(\bar{f}(v) - p)$. Then by the continuity of $\theta(p,q)$ in q, there exists a $q < 1$ such that $n(\theta(p,q))q > n(\theta(p,1)) - \epsilon'$. If so, then $n(\theta(p,q))q[\bar{f}(v) - p] > [n(\theta(p,1)) - \epsilon'][\bar{f}(v) - p] = n(\theta(p,1))[\epsilon + E[\tilde{f}|v] - p] - \epsilon'[\bar{f}(v) - p] = n(\theta(p,1))[E[\tilde{f}|v] - p] + n(\theta(p,1))\epsilon - \epsilon'(\bar{f}(v) - p) = k$, so (p,q) is a profitable deviation, and the beliefs do not support Φ and are therefore not in $\Gamma(\Phi)$. □

Security Design

After the design of the security, but before the sale, the seller receives private information relevant to the payoff of the security. Denote the information by random variable $Z \in \mathbb{R}$, so that the issuer's conditional valuation of the security is $E(F|Z)$. For each security design F, the issuer assumes some liquidity schedule $\theta_F(p,q) : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$; if the seller posts price p, then $\theta_F(p)$ is the market liquidity of the security F. Given a security F, I can write the seller's objective as a function of terms (p,q), and therefore market tightness $\theta_F(p,q)$ as follows:

$$\delta E(X-F|Z) + [1 - m(\theta_F(p,q))]\delta E(F|Z) + m(\theta_F(p,q))[(1-q)\delta E(F|Z) + qp]$$

$$= \delta E(X|Z) + m(\theta_F(p,q))q[p - \delta E(F|Z)].$$

After the seller not only designs the security, but also receives private information z and patience shock δ, then the seller's relevant private information consists of the particular outcome $f \equiv E[F(X)|z]$ of $E(F|Z)$ and the privately known patience. The seller's liquidation problem is

$$\Pi_F(f, \delta) = \max_{p>0, q\in[0,1]} m(\theta_F(p, q))q(p - \delta f). \tag{19}$$

Here, the equilibrium liquidation schedule $\theta_F(p, q)$ and therefore profit function $\Pi_F(f, \delta)$ depend on the structure of the security F. Before receiving private information Z, the seller anticipates this dependency, and designs the security F in order to induce the most favorable profit function $\Pi(f, \delta)$ to maximize his expected profit. Letting $V(F) \equiv E[\Pi_F(E(F|Z))]$ denote the sellers expected profit contingent on security F, the security design problem is

$$\sup_F V(F).$$

Before solving for the optimal F, I first define the following restriction on the conditional distribution of X given Z:

Definition 1. *An outcome $\underline{z}$ of Z is a* uniform worst case *if, for any other outcome z and any interval $I \subset \mathbb{R}_+$ of outcomes of X,*

1. *if $\mu(X \in I|z) > 0$, then $\mu(X \in I|\underline{z}) > 0$;*
2. *the conditional of $\mu(\cdot|z)$ given $X \in I$ has first-order stochastic dominance over the conditional of $\mu(\cdot|\underline{z})$ given $X \in I$.*

Note that the existence of a uniform worst case is weaker than the monotone likelihood ratio property. I am now ready to solve for the optimal security F.

Proof of Proposition 7 Using a strategy identical to the proof of Proposition 10 in DeMarzo and Duffie (1999), I can show that given any increasing security G with $\underline{g} = E[G(X)|\underline{z}]$, then if $F(X) = \min[X, d]$ is a standard debt contract with $\underline{f} = \underline{g}$, then for all z, $g = E[G(X)|z] \geq E[F(X)|z] = f$. The profit function given in (2) depends on the region $[\underline{\delta}, \bar{\delta}]$ and the lower bound

$\underline{f}$, and is strictly decreasing in δf. Of these parameters, ex ante the seller only has control over $\underline{f}$ when designing the security, so write the profit function as $\Pi(\delta f, \underline{f})$. Since $\Pi(\delta f, \underline{f})$ is decreasing in f for any $\delta \in [\underline{\delta}, \bar{\delta}]$, I have $\Pi(\delta g, \underline{g}) = \Pi(\delta g, \underline{f}) \leq \Pi(\delta f, \underline{f})$. Because this inequality holds for any δ and any z, take expectations to get $V(F) \geq V(G)$. So standard debt is an optimal security. □